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SB-3574

M. Sc. - II Examination

March / April – 2011

Advanced Abstract Algebra

(Old Course)

Time : Hours] [Total Marks : **70**

Instructions :

(1)

નીચે દર્શાવેલ ☛ નિશાનીવાળી વિગતો ઉત્તરવહી પર અવશ્ય લખવી. Fillup strictly the details of ☛ signs on your answer book.	Seat No. :
Name of the Examination : ☛ M. Sc. - 2	
Name of the Subject : ☛ Advanced Abstract Algebra - (Old)	
☛ Subject Code No. : 3 5 7 4 ☛ Section No. (1, 2,.....) : Nil	Student's Signature

(2) Answer **all** questions.

(3) **All** questions carry equal marks.

(4) Follow usual notation.

1 (a) If p is a number and $p^m \mid O(G)$ then prove. **5**

that G has a subgroup of order p^m.

(b) Let G be a group. If G is solvable then prove that every subgroup of G and homomorphic image of G are solvable. **5**

(c) Prove that a group of order p^n, where p is prime, is nilpotent. **4**

OR

1 (a) Prove that G is the internal direct product of the normal subgroups $N_1, N_2, \text{-------}, N_n$ if and only if : **5**

(1) $G = N_1 N_2 \ldots\ldots N_n$

(2) $N_i \cap (N_1 N_2 \_\_\_\_\_ N_{i-1} N_{i+1} \_\_\_\_\_\_ N_n) = (e) \, for \; i = 1, 2 \_\_\_\_\_, n.$

(b) If a group of order p^n contains exactly one subgroup **5**

each of orders $p, p^2 \_\_\_\_, p^{n-1}$ then prove that G is cyclic.

(c) Let $H_1, H_2, \_\_\_\_, H_n$ be a family of nilpotent groups. Prove that $H_1 \times H_2 \times \_\_\_\_ \times H_n$ is also nilpotent. 4

2 (a) If an R-module M is generated by a set $\{x_1, x_2, \_\_\_\_\_, x_n\}$ and $1 \in R$ then prove that $M = \sum_{i=1}^{n} Rx_i$. 5

(b) Let V be a nonzero finitely generated vector space over a field F. Prove that V admits a finite basis. 5

(c) Let R be a ring with unity. Let $Hom_R(R, R)$ denote the ring of endomorphism of R regarded as a right R-module Prove that $R \simeq Hom_R(R,R)$ as rings. 4

OR

2 (a) Let M be a simple R-module. Prove that Hom_R (M,M) is a division ring. 5

(b) Let M be an R-module and $RM = \left\{ \sum_i r_i\, m_i \mid r_i \in R, m_i \in M \right\}$. 5

Show that RM is a submodule of M.

(c) Let A and B be R-Submodules of R-modules M and N. respectively. Prove that $\frac{M \times N}{A \times B} \simeq \frac{M}{A} \times \frac{N}{B}$. 4

3 (a) For an R-module M, if M is artinian then every quotient module of M is finitely cogenerated. 5

(b) Let A be a minimal left ideal in a ring R. Prove that either $A^2 = (o)$ or A=Re where e is an idempotent in R. 5

(c) Find the invarient factors of the matrix 4:

$$\begin{bmatrix} -x-3 & 2 & 0 \\ 1 & -x & 1 \\ 1 & -3 & -x-2 \end{bmatrix}$$ over the ring Q[x].

OR

3 (a) Let N be a nil ideal in a noetherian ring R. Prove that N is nilpotent. 5

(b) Find the abelian group generated by $\{x_1, x_2, x_3\}$ subject to 5

$5x_1 + 9x_2 + 5x_3 = 0$

$2x_1 + 4x_2 + 2x_3 = 0$

$x_1 + x_2 - 3x_3 = 0$

(c) Prove that any finite field with p^n elements is the splitting field of $x^{p^n} - x \in F_p[x]$. 4

4 (a) Let $F \subseteq E \subseteq K$ be fields. If $[K:E] < \infty$ and $[E:F] < \infty$ then prove that $[K:F] < \infty$ and $[K:F] = [K:E][E:F]$. 5

(b) Let K and K' be algebraic closures of a field F. Prove that $K \simeq K'$ under an isomorphism that is an identity on F. 5

(c) Show that it is not possible by straightedge and compass alone to trisect $60°$. 4

OR

4 (a) Let P(x) be an irreducible polynominal in F[x]. Prove that there exists an extension.E of F in which P(x) has a root. 5

(b) Let F be the field of rational numbers. Find the splitttting field for the polynomial $x^4 - 2$ over F. 5

(c) Prove that any irreducible polynomial f(x) over a field F of characteristic $P \neq 0$ has multiple roots if and only if there exists $g(x) \in F[x]$ such that $f(x) = g(x^p)$. 4

5 (a) Prove that the multiplicative group of non zero elements of a finite field is cyclic. 5

(b) Construct the Galois group for the polynomial $f(x) = x^4 - x^2 - 2$ over Q. 5

(c) Suppose that the field F contains all the n^{th} roots of unity and $0 \neq a \in F$. Let $x^n - a \in F[x]$ and k be its splitting field over F then prove that (1) K=F(u) (2) the Galois group of $x^n - a$ over F is abelian where u is any root of $x^n - a$. 4

OR

5 (a) Let F be a field of characteristic zero contains all the n^{th} roots of unity for every integer. If $P(x) \in F[x]$ is solvable by redicals over F then prove that Galois group over F of p(x) is a solvable group. 5

(b) Let F be the field of rational numbers and $K = F(2^{1/3})$. Prove that K is not a normal extension of F. 5

(c) Let K be a finite extension of a field F then prove that $O(G(K,F)) \leq [K:F]$. 4
